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Ghost-free Quantization of Gauge Theory on Minimal Fractal Manifolds: Part I

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Abstract

It is known that quantization of massless spin-1 particles runs into several related complications such as the redundancy of gauge orbits, the presence of extra degrees of freedom and the need to introduce "ghost" fields. The textbook interpretation of quantum gauge theory is that "ghosts" are unphysical objects whose function is to preserve Lorentz covariance and unitarity. In particular, Faddev-Popov "ghosts" (FPG) violate the spin-statistics theorem and are devoid of measurable properties. FPG are shown to decouple from the spectrum of observable states, yet it remains unclear how their presence in loop diagrams and their interaction with gauge fields is even possible in the absence of any physical attributes.

The first part of this report is a brief pedagogical review of gauge field quantization. The second part builds up on the idea that, at least in principle, the concept of spacetime endowed with minimal fractality enables a "ghost"-free formulation of quantum gauge theory. An added benefit of this insight is that it sets the stage for a non-perturbative understanding of vacuum polarization in Quantum Electrodynamics.

Key words: Path Integral Quantization, Gauge Theory, Ghost Fields, Faddeev-Popov Method, Gauge Fixing, Minimal Fractal Manifold.

We begin with a brief survey of the main difficulties confronting quantization of abelian and non-abelian fields. The interested reader may consult [ ] for a more comprehensive analysis and additional technical details.

1) Quantization of the Electromagnetic Field

1.a) The classical electromagnetic Lagrangian in the absence of external sources is given by

$$L_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1}$$

where the field strength is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

Maxwell equations read

$$\partial_\mu F^{\mu\nu} = 0 \Rightarrow [\eta_{\mu\nu}(\partial^\rho \partial_\rho) - \partial_\mu \partial_\nu] A^\nu = 0 \tag{3}$$

The Lagrangian (1) is invariant under the group of *local gauge transformations*

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \tag{4}$$

for any function $\Lambda(x)$ satisfying the commutation condition

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\Lambda(x) = [\partial_\mu, \partial_\nu]\Lambda(x) = 0 \tag{6}$$

As a result, the field strength (2) stays unchanged under (4), namely,

$$F_{\mu\nu} \to \partial_\mu (A_\nu + \partial_\nu \Lambda) - \partial_\nu (A_\mu + \partial_\mu \Lambda) = F_{\mu\nu} \tag{7}$$

A fundamental difficulty in quantizing the Maxwell theory is that the second-differential operator

$$\mathrm{D}_{\mu\nu} = \eta_{\mu\nu}(\partial^\rho \partial_\rho) - \partial_\mu \partial_\nu \tag{8}$$

has no inverse as it annihilates any function of the form $\partial_\mu \Lambda(x)$. This implies that, for any given initial data, one cannot uniquely find the potential $A_\mu(x)$ at later times since there is no way of distinguishing between $A_\mu(x)$ and $A_\mu(x) + \partial_\mu \Lambda(x)$. This defines the *redundancy problem of gauge theory*: the phase space of Maxwell's theory is "foliated" by gauge orbits that are inherently over-counted.

1.b) A related difficulty of vector field quantization lies in the number of real components carried by massless spin 1 operators. The electromagnetic potential $A_\mu(x)$ has four independent components, yet the photon has only two independent degrees of freedom called *polarization states*. Let us elaborate on this point with additional details. To examine the plane-wave solutions of Maxwell equations (3), it is customary to consider the momentum space representation of $A_\mu(x)$

$$A_\mu(k) = \frac{1}{(2\pi)^4} \int d^4x A_\mu(x) \exp(-ik \cdot x) \tag{9}$$

Under the gauge transformation, the potential (9) changes as

$$A_\mu(k) \to A_\mu(k) + \Lambda(k) k_\mu \tag{10}$$

Field equations take the form

$$k^2 A_\mu(k) - k_\mu k^\nu A_\nu(k) = 0 \tag{11}$$

and are invariant under (10). One can conveniently resolve $A_\mu(x)$ into four independent vectors, $\varepsilon_\mu(\mathbf{k},\lambda)$, $k_\mu = (\mathbf{k}, k^0)$ and $k^\mu = (\mathbf{k}, -k^0)$, defined by

$$k^\mu \varepsilon_\mu(\mathbf{k},\lambda) = 0 \ , \ \varepsilon_0(\mathbf{k},\lambda) = 0 \qquad (\lambda = 1,2) \tag{12}$$

Hence,

$$A_\mu(k) = a^\lambda(k)\varepsilon_\mu(\mathbf{k},\lambda) + b(k)k_\mu + c(k)k_\mu \tag{13}$$

and the field equations (11) turn into

$$k^2 a^\lambda(k)\,\varepsilon_\mu(\mathbf{k},\lambda) + b(k)[k^2 - (k\cdot k)k_\mu] = 0\,, \ \ (k\cdot k) > 0 \tag{14}$$

which forces the coefficient functions to vanish, namely,

$$k^2 a^\lambda(k) = 0, \quad (\lambda = 1,2) \tag{13a}$$

$$b(k) = 0 \tag{13b}$$

Relations (13) show that the field equations cannot fix the value of the coefficient $c(k)$. This implies that $c(k)$ can be set to zero by means of a suitable gauge transformation, which, in turn, means that $c(k)$ has no physical meaning. One arrives at the conclusion that there are only two independent plane wave solutions on the light cone ($k^2 = 0$) and *two transverse polarization vectors*.

The standard solution to the gauge redundancy problem of Maxwell theory is *gauge fixing*. The method reduces the number of allowed orbits to a smaller set, where all the orbits are related by smaller gauge group symmetry. Since quantum gauge theory is often described using the path-integral (PI) formulation, a generalization of gauge fixing to non-abelian fields is required to ensure internal consistency of the theory. This is the motivation for the Faddeev-Popov (FP) method described in the next paragraph.

1.c) Unlike the case of massive fields, the spin of a massless particle cannot be defined relative to its rest frame of reference. As a result, the three-dimensional rotation group is no longer adequate for characterizing the photon spin and it is replaced by the group of two-dimensional rotations around the three-momentum vector $\mathbf{k}$ [ ]. The reality of only two transverse photon polarizations hints to a violation of Lorentz invariance stemming from the fact that transversality is not preserved by Lorentz transformations. It can be shown, however, that Lorentz symmetry is restored provided that photons couple to *conserved currents* defined through $\partial_\mu J^\mu = 0$. The existence of such currents is a direct consequence of gauge invariance.

<u>2) The Faddeev-Popov method</u>

The FP method consists in applying a suitable constraint in the PI description of gauge theory that automatically removes the ambiguity associated with the gauge transformation. Consider the generating functional

$$Z[J] = \int DA_\mu \exp i \int d^4x \,(L + J^\mu A_\mu) \tag{14}$$

The integral measure $DA_\mu = \Pi_x \, dA_\mu(x)$ spans over all possible vector potentials A_μ and necessarily includes their gauge transforms (4). Explicitly writing (4) as

$$A_\mu = \overline{A_\mu} + \partial_\mu \Lambda(x) \tag{15}$$

factors out the contribution of $\overline{A_\mu}$ and $\Lambda(x)$ in (14), namely,

$$Z[J] = \int D\overline{A_\mu} \exp i \int d^4x \, (L + J^\mu \overline{A_\mu}) \int D\Lambda \tag{16}$$

The presence of the second integral over the arbitrary field $\Lambda(x)$ causes the generating functional to diverge since there are unaccountable many $\Lambda(x)$ contributing to (16). Following the FP method [ ], the generating functional (16) is cast in the equivalent form

$$Z[J] = \int D\Lambda \, F[A_\mu, \Lambda; J] \, \Delta[A_\mu] \, \delta(G[A^\Lambda]) \tag{17}$$

where

$$F[A_\mu, \Lambda; J] = \int dA_\mu \exp i \int d^4x (L + J^\mu A_\mu) \tag{18}$$

and

$$G[A^\Lambda] = \partial^\mu A_\mu \tag{19}$$

The FP determinant is defined as

$$\Delta[A] = \det\left(\frac{\delta G[A^\Lambda]}{\delta \Lambda}\right), \quad \Lambda = 0 \tag{20}$$

and leads to the introduction of “ghost” and “anti-ghost” fields. In particular, the “ghost” part of the Lagrangian in Yang-Mills theory is given by [ ]

$$L_g = \partial_\mu \overline{c^a} \partial^\mu c^a + g f^{abc} (\partial^\mu \overline{c^a}) A_b^\mu c^c \qquad (21)$$

Here, "a" is the index of the gauge group,"g" stands for the coupling charge and "f^{abc}" for the structure constants. The first term is the kinetic component of the Lagrangian containing the contribution of “ghosts”(c^a) and their antiparticles ($\overline{c^a}$), whereas the second term reflects the interaction of ghosts with the gauge field. In Yang-Mills theory, “ghosts” violate the spin-statistics theorem in that they are spinless complex scalar fields with fermion statistics [ ].